

Graded Fock-like representations for a system of algebraically interacting paraparticles

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Abstract. We will present and study an algebra describing a mixed paraparticle model, known in the bibliography as “The Relative Parabose Set (RPBS)”. Focusing in the special case of a single parabosonic and a single parafermionic degree of freedom $P_{BF}^{(1,1)}$, we will construct a class of Fock-like representations of this algebra, dependent on a positive parameter p a kind of *generalized parastatistics order*. Mathematical properties of the Fock-like modules will be investigated for all values of p and constructions such as ladder operators, irreducibility (for the carrier spaces) and $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -gradings (for both the carrier spaces and the algebra itself) will be established.

1. Introduction

Our central object of study in this short letter, will be the *Relative Parabose Set algebra* $P_{BF}^{(1,1)}$ in a single parabosonic and a single parafermionic degree of freedom. It belongs to the general family of paraparticle algebras intimately related to the Wigner quantization scheme [7]. The “free” paraparticle algebras (parabosonic and parafermionic algebras) have been -implicitly- introduced in the early '50's [2] while the “mixed” paraparticle models such as P_{BF} (and a couple of others as well) have been first introduced in [3].

The algebra $P_{BF}^{(1,1)}$ is generated (as an algebra over \mathbb{C}) by the four generators b^+, b^-, f^+, f^- subject to the usual trilinear relations of the *free parabosonic* and the *free parafermionic* algebras which can be compactly summarized as

$$[\{b^\xi, b^\eta\}, b^\epsilon] = (\epsilon - \eta)b^\xi + (\epsilon - \xi)b^\eta \quad , \quad [[f^\xi, f^\eta], f^\epsilon] = \frac{1}{2}(\epsilon - \eta)^2 f^\xi - \frac{1}{2}(\epsilon - \xi)^2 f^\eta \quad (1)$$

for all values $\xi, \eta, \epsilon = \pm$, together with the *mixed trilinear relations*

$$[\{b^\xi, b^\eta\}, f^\epsilon] = [[f^\xi, f^\eta], b^\epsilon] = 0, \quad [\{f^\xi, b^\eta\}, b^\epsilon] = (\epsilon - \eta)f^\xi, \quad \{\{b^\xi, f^\eta\}, f^\epsilon\} = \frac{1}{2}(\epsilon - \eta)^2 b^\xi \quad (2)$$

for all values $\xi, \eta, \epsilon = \pm$, which represent a kind of algebraically established interaction between parabosonic and parafermionic degrees of freedom and *characterize the relative parabose set*.

It is easy for one to see that when all combinations of the $\xi, \eta, \epsilon = \pm$ are taken into account, the first of equations (1) produces 6 relations, the second of eq. (1) produces 2 relations and equations (2) produce 24 relations. One can easily see (see also the discussion in [5]) that not

all of these $24 + 6 + 2 = 32$ are algebraically independent. However, we keep the relations in the form given in equations (1), (2) and we do not proceed in further simplifying them, because of the compact notational and computational advantages of this presentation. (See also [4, 5, 8]).

The purpose of this letter, will be to present the construction of a class of representations for the relations (1), (2). We will present an infinite class of irreducible, $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -graded representations of these relations. This class will be parametrized by the values of a positive integer p (similarly to the way that the free parabosonic and parafermionic algebra Fock-like spaces are parametrized by the values of a positive integer). The parameter p will be called *generalized parastatistics order*. It will be shown that the b^\pm, f^\pm generators act as “ladder” operators (*Creation/Annihilation* operators) in a kind of two-dimensional “ladder”, generalizing thus the way that the usual Canonical Commutation–Anticommutation relations (CCR–CAR) are represented in their Fock spaces and also the way that the single degree of freedom free parabosonic and free parafermionic algebras act in their own Fock-spaces [6] respectively.

2. Structure of the Fock-like spaces

In this section we are going to briefly review the results of [9] on the construction of the Fock-like spaces, which will serve as carrier spaces for the Fock-like representations of $P_{BF}^{(1,1)}$.

An old conjecture [3] on the study of the representations of the paraparticle algebras, states that if we consider representations of $P_{BF}^{(1,1)}$, satisfying the adjointness conditions $(b^-)^\dagger = b^+$ and $(f^-)^\dagger = f^+$, on a complex, infinite dimensional, pre-Hilbert¹ space, possessing a unique vacuum vector $|0\rangle$ satisfying $b^-|0\rangle = f^-|0\rangle = b^-f^+|0\rangle = f^-b^+|0\rangle = 0$ and $\langle 0|0\rangle = 1$ then the following additional conditions (where p may be an arbitrary positive integer)

$$b^-b^+|0\rangle = f^-f^+|0\rangle = p|0\rangle \quad (3)$$

single out an irreducible representation which is unique up to unitary equivalence. In other words the above statement, tells us that for any positive integer p there is an irreducible representation of $P_{BF}^{(1,1)}$ uniquely specified (up to unitary equivalence) by the above relations.

We now proceed in summarizing the results of [9]: The carrier spaces of the Fock-like representations of $P_{BF}^{(1,1)}$ are $\bigoplus_{n=0}^p \bigoplus_{m=0}^\infty \mathcal{V}_{m,n}$ where p is an arbitrary (but fixed) positive integer. The subspaces $\mathcal{V}_{m,n}$ are 2-dim except for the cases $m = 0, n = 0, p$ i.e. except the subspaces $\mathcal{V}_{0,n}, \mathcal{V}_{m,0}, \mathcal{V}_{m,p}$ which are 1-dim. Let us see how the corresponding vectors look like:

- If $0 < m$ and $0 < n < p$, then the subspace $\mathcal{V}_{m,n}$ is spanned by all vectors of the form

$$\left| \begin{matrix} m_1, m_2, \dots, m_l \\ n_0, n_1, n_2, \dots, n_l \end{matrix} \right\rangle \equiv (f^+)^{n_0} (b^+)^{m_1} (f^+)^{n_1} (b^+)^{m_2} (f^+)^{n_2} \dots (b^+)^{m_l} (f^+)^{n_l} |0\rangle \quad (4)$$

where $m_1 + m_2 + \dots + m_l = m$, $n_0 + n_1 + n_2 + \dots + n_l = n$ and $m_i \geq 1$ (for $i = 1, 2, \dots, l$), $n_i \geq 1$ (for $i = 1, 2, \dots, l-1$) and $n_0, n_l \geq 0$.

For any specific combination of values (m, n) the corresponding subspace $\mathcal{V}_{m,n}$ has a basis consisting of the two vectors ($R^\eta = \frac{1}{2}\{b^\eta, f^\eta\}$ for $\eta = \pm$)

$$|m, n, \alpha\rangle \equiv (f^+)^n (b^+)^m |0\rangle = \left| \begin{matrix} m \\ n, 0 \end{matrix} \right\rangle, \quad |m, n, \beta\rangle \equiv (f^+)^{(n-1)} (b^+)^{(m-1)} R^+ |0\rangle = \frac{1}{2} \left| \begin{matrix} m \\ n-1, 1 \end{matrix} \right\rangle + \frac{1}{2} \left| \begin{matrix} m-1, 1 \\ n-1, 1, 0 \end{matrix} \right\rangle \quad (5)$$

- If $m = 0$ or $n = 0, p$, the vectors $|0, n, \beta\rangle$ and $|m, 0, \beta\rangle$ are (by definition) zero and furthermore, for $m \neq 0$, the vector $|m, p, \beta\rangle$ becomes parallel to $|m, p, \alpha\rangle$: $|m, p, \beta\rangle = \frac{1}{p} |m, p, \alpha\rangle$
- If $n \geq p + 1$, all basis vectors of (5) vanish.

¹ in the sense that it is an inner product space, but not necessarily complete with respect to the inner product.

Note 1: The above described subspaces of $\bigoplus_{n=0}^p \bigoplus_{m=0}^{\infty} \mathcal{V}_{m,n}$ can be visualized as follows:

$$\begin{array}{ccccccc}
\mathcal{V}_{0,0} & \mathcal{V}_{0,1} & \dots & \mathcal{V}_{0,n} & \dots & \dots & \mathcal{V}_{0,p-1} & \mathcal{V}_{0,p} \\
\mathcal{V}_{1,0} & \mathcal{V}_{1,1} & \dots & \mathcal{V}_{1,n} & \dots & \dots & \mathcal{V}_{1,p-1} & \mathcal{V}_{1,p} \\
\vdots & \vdots & \dots & \vdots & \dots & \dots & \vdots & \vdots \\
\mathcal{V}_{m,0} & \mathcal{V}_{m,1} & \dots & \mathcal{V}_{m,n} & \mathcal{V}_{m,n+1} & \dots & \dots & \mathcal{V}_{m,p} \\
\vdots & \vdots & \dots & \mathcal{V}_{m+1,n} & \dots & \dots & \vdots & \vdots \\
\vdots & \vdots & \dots & \vdots & \dots & \dots & \vdots & \vdots
\end{array}$$

• Inside each one of the above presented 2-dim subspaces $\mathcal{V}_{m,n}$ (with $m \neq 0$, $n \neq 0, p$) the vectors of (5) are linearly independent and constitute a basis. However, these vectors are neither orthogonal nor normalized. We can show that an orthonormal set of basis vectors can be obtained as

$$|m, n, +\rangle = c_+ |m, n, \alpha\rangle \quad |m, n, -\rangle = -c_- \left(|m, n, \alpha\rangle - p |m, n, \beta\rangle \right)$$

where c_{\pm} are suitable normalization factors [9]. Now we can show orthonormalization $\langle m, n, s | m', n', s' \rangle = \delta_{m,m'} \delta_{n,n'} \delta_{s,s'}$ and completeness $\sum_{m=0}^{\infty} \sum_{n=0}^p \sum_{s=\pm} |m, n, s\rangle \langle m, n, s| = 1$.

Note 2: If we consider the Hermitian operators $N_s = \frac{1}{p}(N_f^2 - (p+1)N_f + f^+ f^- + \frac{p}{2})$, $N_f = \frac{1}{2}[f^+, f^-] + \frac{p}{2}$ and $N_b = \frac{1}{2}\{b^+, b^-\} - \frac{p}{2}$ we can show that they constitute a *Complete Set of Commuting Observables* (C.S.C.O.): We have $[N_b, N_f] = [N_b, N_s] = [N_f, N_s] = 0$; their common eigenvectors are exactly the elements of the orthonormal basis formerly described. Any vector $|m, n, s\rangle$ is uniquely determined as an eigenvector of N_b, N_f, N_s by its eigenvalues $0 \leq m$, $0 \leq n \leq p$ and $s = \pm \frac{1}{2}$ respectively.

3. Main results: Construction of the Fock-like representations

For detailed computations and proofs of the results presented in this section see [5].

3.1. Construction of ladder operators

We now present the formulae describing explicitly the action of the generators (and hence of the whole algebra) of $P_{BF}^{(1,1)}$ on the carrier spaces $\bigoplus_{n=0}^p \bigoplus_{m=0}^{\infty} \mathcal{V}_{m,n}$ for any positive integer p :

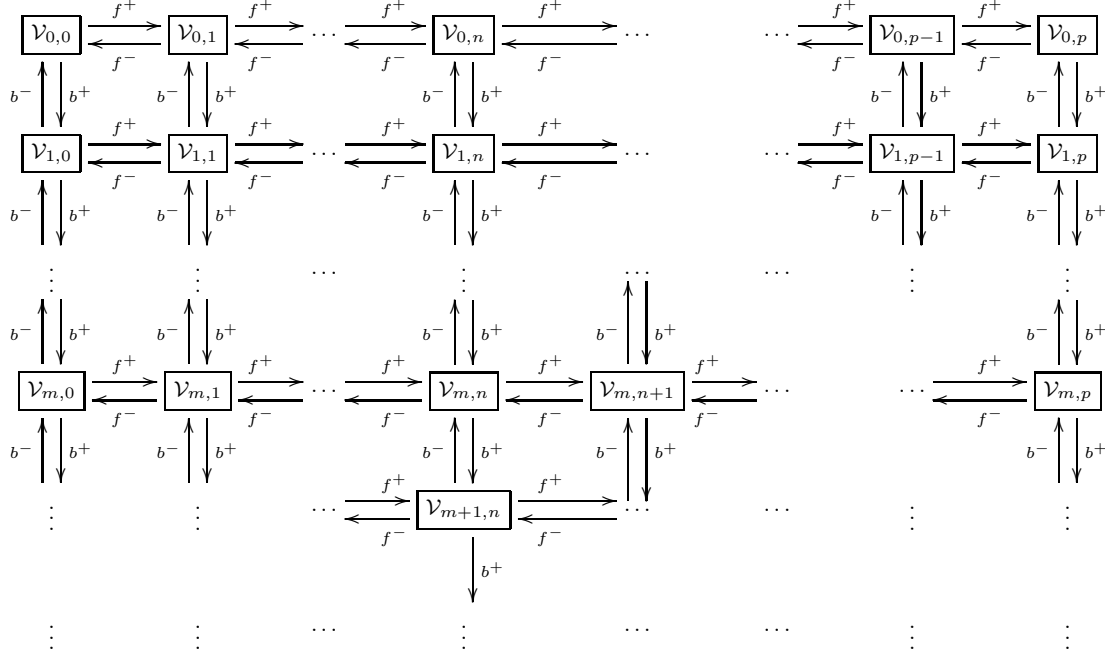
$$\begin{aligned}
\nabla b^- \cdot |m, n, \alpha\rangle &= \begin{cases} (-1)^n m |m-1, n, \alpha\rangle - 2(-1)^n n m |m-1, n, \beta\rangle, & \underline{m : even} \\ -(-1)^n (2n - m - (p-1)) |m-1, n, \alpha\rangle - 2(-1)^n n (m-1) |m-1, n, \beta\rangle, & \underline{m : odd} \end{cases} \\
\nabla b^- \cdot |m, n, \beta\rangle &= \begin{cases} -(-1)^n |m-1, n, \alpha\rangle + (-1)^n (2n - m - p) |m-1, n, \beta\rangle, & \underline{m : even} \\ -(-1)^n |m-1, n, \alpha\rangle - (-1)^n (m-1) |m-1, n, \beta\rangle, & \underline{m : odd} \end{cases} \\
\nabla f^- \cdot |m, n, \alpha\rangle &= n(p+1-n) |m, n-1, \alpha\rangle, \quad \nabla b^+ \cdot |m, n, \alpha\rangle = (-1)^n |m+1, n, \alpha\rangle - (-1)^n 2n |m+1, n, \beta\rangle \\
\nabla f^- \cdot |m, n, \beta\rangle &= |m, n-1, \alpha\rangle + (n-1)(p-n) |m, n-1, \beta\rangle, \quad \nabla b^+ \cdot |m, n, \beta\rangle = -(-1)^n |m+1, n, \beta\rangle \\
\nabla f^+ \cdot |m, n, \alpha\rangle &= \begin{cases} |m, n+1, \alpha\rangle, & \underline{if \ n \leq p-1} \\ 0, & \underline{if \ n \geq p} \end{cases} \quad \nabla f^+ \cdot |m, n, \beta\rangle = \begin{cases} |m, n+1, \beta\rangle, & \underline{if \ n \leq p-1} \\ 0, & \underline{if \ n \geq p} \end{cases}
\end{aligned} \tag{6}$$

for all integers $0 \leq m$, $0 \leq n \leq p$. The direct proof [5] of these formulae involves lengthy “normal-ordering” algebraic computations inside $P_{BF}^{(1,1)}$. We must take into account: (a). the relations (1), (2) of $P_{BF}^{(1,1)}$, (b). the relations (3) together with $b^-|0\rangle = f^-|0\rangle = b^-f^+|0\rangle = f^-b^+|0\rangle = 0$ and (c). the structure and properties of the corresponding carrier space as described in Sect. 2.

Apart from the direct proof, eq. (6) have also been checked and verified via the Quantum [1] add-on for Mathematica 7.0, which is an add-on for performing symbolic algebraic computations, including the use of generalized Dirac notation. What we have actually verified via the use of this package, is that the actions formulae (6) preserve all of the relations (1), (2) of $P_{BF}^{(1,1)}$.

3.2. Irreducibility

In the following diagram we provide a “visual” interpretation of relations (6) i.e. of the action of the generators of $P_{BF}^{(1,1)}$ on the direct summand subspaces of the Fock-like carrier space $\bigoplus_{n=0}^p \bigoplus_{m=0}^{\infty} \mathcal{V}_{m,n}$. We can easily figure out that we have a kind of *generalized creation-annihilation operators* acting on a two dimensional ladder of subspaces:



Initiating from the remark that we are dealing with a cyclic module which moreover can be generated by any of its elements, we can prove [5] that the Fock-like representations, explicitly given by (6) and visually represented in the above diagram, are irreducible representations (or: simple $P_{BF}^{(1,1)}$ -modules) for any $p \in \mathbb{N}^*$.

3.3. Klein-group Grading of the representation

Defining $\deg|m, n, \alpha\rangle = \deg|m, n, \beta\rangle = (m \bmod 2, n \bmod 2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ for the carrier spaces and $\deg b^\pm = (1, 0)$, $\deg f^\pm = (0, 1)$ for the algebra, the Fock-like representations of $P_{BF}^{(1,1)}$ over $\bigoplus_{n=0}^p \bigoplus_{m=0}^{\infty} \mathcal{V}_{m,n}$, become $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -graded modules, $\forall p \in \mathbb{N}^*$. (For proof and details see [5]).

Acknowledgments

KK would like to thank the whole staff of IFM, UMSNH for providing a challenging and stimulating atmosphere while preparing this article. His work was supported by the research project CONACYT/No. J60060. AHA was supported by CIC 4.16 and CONACYT/No. J60060; he is also grateful to SNI.

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